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Hawking Radiation via Tunnelling from Black Holes by Using Eddington–Finkelstein Coordinates

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In this paper, by using well-known Eddington–Finkelstein coordinates instead of Painlevè coordinates, we study the tunnelling effect of the black holes once again. As examples of the static and stationary black holes, we calculate the tunnelling rates of Schwarzschild and Kerr black holes. In addition, the result obtained by adopting Eddington–Finkelstein coordinates is in agreement with the Parikh's and Zhang's recent work which adopts the Painlevè coordinates. At last, we discuss carefully the condition that the coordinates system in which we study the tunnelling process should satisfy. In our opinion, the terms of the tunnelling effect are not as strict as ones in Parikh's paper and could be softened properly.

KEY WORDS: Eddington–Finkelstein coordinates; Tunnelling effect; WKB approximation.

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1. INTRODUCTION

Over 30 years ago, Stephen Hawking discovered that basic principles of quantum field theory lead to the emission of thermal radiation from a classical black hole (Hawking, 1975), which gives rise to a famous paradox—the information loss paradox of black hole physics. Recently, Parikh and Wilczek treated Hawking radiation as a tunnelling process in order to solve the information loss paradox (Parikh, 2004a, 2004b; Parikh and Wilczek, 2000). They obtained a leading correction to the emission rate arising from loss of mass of the black hole. Following this method, Zhang and Zhao extend the investigation to stationary axisymmetric Kerr black holes and the result is successful (Zhang and Zhao, 2005). However, there are two difficulties to overcome in the calculation of the tunnelling rate. The first is that there do not seem to be any barrier. The second is that in order to

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do a tunnelling computation, one requires to find a coordinate system which is well-behaved at the event horizon.

To overcome the first difficulty, they think that the barrier is created by the outgoing particle itself and energy (ADM energy) must be conserved. As the black hole radiates, it loses energy. Because the energy and radius of the event horizon are related, the black hole has to shrink. It is this constriction that sets the scale: the horizon recedes from its original radius to a new, smaller radius. Moreover, the amount of constriction is related to the energy of the outgoing particle. So, there is no pre-existing barrier and it is the tunnelling particle itself that defines the barrier (Parikh and Wilczek, 2000).

In order to overcome the second difficulty, Parikh introduced the Painlevè coordinates (1921), $t = t_s + 2\sqrt{2mr} + 2m \ln \frac{\sqrt{r}-\sqrt{2m}}{\sqrt{r}+\sqrt{2m}}$, where t_s is Schwarzschild time and t is Painlevè time. The Painlevè metric has many attractive features. Firstly, the components of the metric in Painlevè coordinates are regular, not diverging at the event horizon. Secondly, constant-time slices are just flat Euclidean space. In paper (Parikh, 2004a), this feature is rather important because the WKB approximation is applied to calculate the tunnelling rate. WKB approximation is derived from the quantum mechanics which is right in flat space. Thirdly, there exists a time-like killing vector field, which is important to the energy conservation. Finally, because in quantum mechanics, particle tunnelling a barrier is a instantaneous process, Zhao and Zhang (2005) suggest that the metric in the coordinates should satisfy Landau coordinate clock synchronization condition (Landau and Lifshitz, 1975). Fortunately, the metric in the Painlevè coordinates does satisfy Landau coordinate clock synchronization condition.

In our opinion, the main aim and the crucial point to introduce a new coordinate system is to eliminate the singularity of the components of the metric at the event horizon. It is not only the Painlevè coordinates which satisfy the condition above. The well-known Eddington–Finkelstein coordinate, $v = t + r_*$, is rather suitable to study the tunnelling effect too, where r_* is the tortoise coordinate. When we use the Eddington–Finkelstein coordinates instead of the Schwarzschild coordinates, the components of the metric are not singular too. So, we believe that we could study the tunnelling process adopting Eddington–Finkelstein coordinates. In this paper, using the Eddington–Finkelstein coordinate, we calculate the tunnelling rate of the Schwarzschild and the Kerr black holes. The results are rather successful and in agreement with Parikh (2004a) and Zhang and Zhao (2005) results, respectively. Eddington–Finkelstein coordinates have some merits too. The line elements of the black holes in Eddington–Finkelstein coordinates are more simple than the line elements in Painlevè coordinates, which make us calculate the tunnelling rate more easily. It is interesting that the constant-time slices in Eddington–Finkelstein coordinates are not flat Euclidean space. According to the condition that the constant-time slices should be flat Euclidean space

(Parikh, 2004b), it seems that the WKB approximation could not be used to study the tunnelling process in Eddington–Finkelstein coordinates. We, however, will calculate the tunnelling rate of both the Schwarzschild and Kerr black holes and obtain the correct results. The reasonable explanation is that the WKB approximation could be extended to the space which is not flat Euclidean, although the WKB approximation is derived from the quantum mechanics. So, in our opinion, the conditions which the coordinate system should satisfy in Parikh's paper is too strict and could be softened properly. In addition, another important condition that the coordinates system should satisfy is that the event horizon and the time-like limit surface should coincide because we use WKB formula when we calculate the tunnelling rate. The WKB approximation could be used only when the language of a point particle is appropriate at the event horizon. Because the infinite blue-shift takes place near the time-like limit surface, the characteristic wavelength of any wave packet is always arbitrarily small there and the geometrical optics limit becomes an especially reliable approximation. So, the event horizon and the time-like limit surface should be identical as we study the tunnelling effect.

The paper is organized as follows. In Sections 2 and 3, we calculate the tunnelling rate of the Schwarzschild and the Kerr black holes, respectively. Finally, we will discuss the conditions of the coordinates which could be used to study the tunnelling rate. Throughout the paper, the units $G = c = \hbar = k_B = 1$ are used.

2. TUNNELLING EFFECT FROM SCHWARZSCHILD BLACK HOLES

To describe tunnelling process, it is necessary to choose coordinates which are not singular at the event horizon. We choose Eddington–Finkelstein coordinate, $v = t + r_*$, not the Painlevè coordinates, to study the tunnelling effect, where $r_* = r + 2m\ln(\frac{r}{2m} - 1)$ is tortoise coordinate. With this choice, the line element reads

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\varphi^2) \tag{1}$$

Comparing with the line elements in Painlevè coordinates (1921)

$$ds^2 = -\left(1 - \frac{2m}{r}\right)dt^2 + 2\sqrt{\frac{2m}{r}}dtdr + dr^2 + r^2d\Omega^2, \tag{2}$$

the line elements (1) are more simple, which make the calculation more easily. It is obvious that the components are not diverging at the event horizon in Eq. (1) and $\left(\frac{\partial}{\partial v}\right)^a$ is a time-like vector field. One distinction between these two kinds of the coordinates is that the constant-time slices of the line elements in the Eddington–Finkelstein coordinates are not flat Euclidean space. The calculation below will show that this is not important to study the tunnelling process.

The radial null geodesics in Eddington–Finkelstein coordinates obey

$$\dot{r} \equiv \frac{dr}{dv} = \frac{1}{2}\left(1 - \frac{2m}{r}\right) \tag{3}$$

Equations (1) and (3) are modified when the self-gravitation of the particle is considered. We could consider the particle with energy ω as a shell of energy. We fix the total mass (ADM mass) and allow the hole mass to fluctuate. When the shell of energy ω travels on the geodesics, we should replace m with $m - \omega$ in the geodesic Eq. (3) and in the line elements Eq. (1) to describe the moving of the shell (Parikh and Wilczek, 2000).

In our picture, a point particle description is appropriate. Because of the infinite blue shift near the horizon, the characteristic wavelength of any wave packet is always arbitrarily small there, so that the geometrical optics limit becomes an especially reliable approximation. The geometrical limit allows us to obtain rigorous results directly in the language of particles, rather than having to use the second-quantized Bogolubov method. In fact, the point particle description here demands that the event horizon and the time-like limit surface coincide because the infinite blue shift occur only near the time-like limit surface. Fortunately, the two surfaces are identical to the Schwarzschild black hole in Eddington–Finkelstein coordinates. In the semiclassical limit, we could apply the WKB formula. This relates the tunnelling amplitude to the imaginary part of the particle action at stationary phase. The emission rate, Γ, is the square of the tunnelling amplitude (Parikh, 2004b):

$$\Gamma \sim \exp(-2\,\mathrm{Im}S) \tag{4}$$

The imaginary part of the action for an outgoing positive energy particle which crosses the horizon outwards from r_{in} to r_{out} could be expressed as

$$\mathrm{Im}S = \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} p_r\, dr = \mathrm{Im}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}}\int_0^{p_r} dp'_r\, dr \tag{5}$$

where p_r is canonical momentum conjugate to r, $r_{\mathrm{in}} = 2m$ is the initial radius of the black hole, and $r_{\mathrm{out}} = 2(m - \omega)$ is the final radius of the hole. We substitute Hamilton equation $\dot{r} = \frac{dH}{dp_r}\,|_r$ into Eq. (5), change variable from momentum to energy, and switch the order of integration to obtain

$$\mathrm{Im}S = \mathrm{Im}\int_m^{m-\omega}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \frac{dr}{\dot{r}}\, dH = \mathrm{Im}\int_0^{\omega}\int_{r_{\mathrm{in}}}^{r_{\mathrm{out}}} \frac{2dr}{1 - \frac{2(m-\omega')}{r}}(-d\omega') \tag{6}$$

We have used the modified Eq. (3) and H is the ADM energy of the space-time (Parikh, 2004b). In Eq. (6), $r = r'_H = m - \omega'$ is the first-order pole. Now the integral can be done by deforming the contour, so as to ensure that positive energy solutions decay in time (that is , into the lower half ω' plane) (Parikh, 2004b). In this way we obtain

$$\mathrm{Im}S = 4\pi\omega\left(m - \frac{\omega}{2}\right) \tag{7}$$

The tunnelling rate is therefore

$$\Gamma \sim \exp\left(-2\,\mathrm{Im}S\right) = \exp\left[-8\pi m\omega\left(1 - \frac{\omega}{2m}\right)\right] = \exp\left(\triangle S_{\mathrm{BH}}\right) \tag{8}$$

where S_{BH} is the Bekenstein–Hawking entropy. This result is in agreement with Parikh's work. Although the constant-time slices are not flat Euclidean space, using the WKB approximation, we obtain the correct result. This shows that WKB approximation could be extended to the space which is not flat Euclidean.

3. TUNNELLING EFFECT FROM KERR BLACK HOLES

The line element of the non-stationary Kerr black hole in Eddington–Finkelstein coordinates could be written as (Carmeli, 1982)

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2m\left(v\right)r}{\rho^2}\right)dv^2 + 2dvdr - \frac{4m\left(v\right)ra\sin^2\theta}{\rho^2}dvd\varphi + \rho^2 d\theta^2 \\ &\quad -2a\sin^2\theta\, drd\varphi + \left[(r^2 + a^2) + \frac{2m\left(v\right)ra^2\sin^2\theta}{\rho^2}\right]\sin^2\theta\, d\varphi^2 \end{aligned} \tag{9}$$

If m is a constant, not the function of time v, the metric (9) could return to the stationary Kerr black hole in Eddington–Finkelstein coordinates

$$\begin{aligned} ds^2 &= -\left(1 - \frac{2mr}{\rho^2}\right)dv^2 + 2dvdr - \frac{4mra\sin^2\theta}{\rho^2}\,dvd\varphi + \rho^2 d\theta^2 \\ &\quad -2a\sin^2\theta\, drd\varphi + \left[(r^2 + a^2) + \frac{2mra^2\sin^2\theta}{\rho^2}\right]\sin^2\theta\, d\varphi^2 \qquad (10) \\ &= g_{00}\,dv^2 + 2dvdr + 2g_{03}\,dvd\varphi + \rho^2\,d\theta^2 + 2g_{13}\,drd\varphi + g_{33}\,d\varphi^2 \end{aligned}$$

where

$$\rho^2 = r^2 + a^2\cos^2\theta \tag{11}$$

$$\Delta = r^2 + a^2 - 2mr \tag{12}$$

We will study the tunnelling effect in this coordinates; Zhang and Zhao (2005) studied it in Painlevè coordinates before. Although the components of the metric in Eq. (10) are not singular, we have to introduce the dragging coordinate system. The reason is that the event horizon does not coincide with the time-like limit surface. Let

$$\frac{d\varphi}{dv} = -\frac{g_{03}}{g_{33}} = \Omega \tag{13}$$

The line element of Kerr black hole can be rewritten as

$$\begin{aligned} ds^2 &= -\frac{\rho^2\Delta}{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}\,dv^2 \\ &\quad + \frac{2[(r^2+a^2)^2 - \Delta a^2\sin^2\theta - 2a^2mr\sin^2\theta]}{(r^2+a^2)^2 - \Delta a^2\sin^2\theta}\,dv\,dr + \rho^2 d\theta^2 \\ &= \tilde{g}_{00}\,dv^2 + 2\tilde{g}_{01}\,dv\,dr + \rho^2 d\theta^2 \end{aligned} \tag{14}$$

In line element (14), from $\tilde{g}_{00} = 0$, we could get $r = r_+$, where r_+ is the radius of the event horizon. The radial null geodesics could be written as

$$\dot{r} = \frac{dr}{dv} = \frac{\rho^2\Delta}{2[(r^2+a^2)^2 - \Delta a^2\sin^2\theta - 2a^2mr\sin^2\theta]} \tag{15}$$

When a particle of energy ω and angular momentum ωa tunnels out, the mass and the angular momentum of the black hole will become to $m - \omega$ and $(m-\omega)\,a$. So, we should replace m with $m - \omega$ in the line element (14) and the geodesic Eq. (15). The imaginary part of the action could be written as (Zhang and Zhao, 2005)

$$\operatorname{Im}S = \operatorname{Im}\left[\int_{r_{\text{in}}}^{r_{\text{out}}}\int_0^{P_r} P_r'\,dr - \int_{\varphi_{\text{in}}}^{\varphi_{\text{out}}}\int_0^{P_\varphi} P_\varphi'\,d\varphi\right] \tag{16}$$

where P_r and P_φ are two canonical momentum conjugates to r and φ, respectively. Applying the Hamilton equation, we could get

$$\dot{r} = \frac{dH}{dP_r} = -\frac{d\omega}{dP_r} \tag{17}$$

$$\dot{\varphi} = \frac{dH}{dP_\varphi} = \frac{\Omega_H\,dJ}{dP_\varphi} = -a\Omega_H\frac{d\omega}{dP_\varphi} \tag{18}$$

where $\Omega_H = -\frac{g_{03}}{g_{33}}\,|_{r=r_+}$. $dH = \Omega_H\,dJ$ represents energy changes of hole due to the loss of the angular momentum when a particle tunnels out. So, we could get

$$\operatorname{Im}S = \operatorname{Im}\left[\int_0^{\omega}\int_{r_{\text{in}}}^{r_{\text{out}}}\frac{dr}{\dot{r}'}d(-\omega') - \int_0^{\omega}\int_{\varphi_{\text{in}}}^{\varphi_{\text{out}}}\frac{d\varphi}{\dot{\varphi}'}a\Omega_H'\,d(-\omega')\right] \tag{19}$$

From $\dot{\varphi} = \frac{d\varphi}{dv}$ and $\dot{r} = \frac{dr}{dv}$, we could obtain $\frac{d\varphi}{\dot{\varphi}} = \frac{dr}{\dot{r}}$. Eventually, Eq. (19) becomes to

$$\mathrm{Im}S = \mathrm{Im}\left[\int_0^{\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}'} d(-\omega') - \int_0^{\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{dr}{\dot{r}'} a\Omega'_H\, d(-\omega')\right] \tag{20}$$

where $\dot{r}' \equiv \dot{r}\ (m-\omega')$ and $\Omega'_H \equiv \Omega_H(m-\omega')$. Substituting $\dot{r}'$ and Ω'_H into the Eq. (20), we could obtain

$$\mathrm{Im}S = \left\{\mathrm{Im}\int_0^{\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{2[(r^2+a^2)^2 - \Delta' a^2 \sin^2\theta - 2a^2(m-\omega')r\sin^2\theta]}{\rho^2\Delta'} dr(-d\omega')\right. \tag{21}$$

$$\left. - \int_0^{\omega}\int_{r_{\rm in}}^{r_{\rm out}} \frac{2[(r^2+a^2)^2 - \Delta' a^2\sin^2\theta - 2a^2(m-\omega')r\sin^2\theta]a[2(m-\omega')ra]}{\rho^2\Delta'[\rho^2(r^2+a^2) + 2a^2(m-\omega')r\sin^2\theta]}\, dr(-d\omega')\right\} \tag{22}$$

where $\Delta' \equiv r^2 + a^2 - 2(m-\omega')r = (r - r'_+)(r - r'_-)$ and $r'_+ \equiv r_+(m-\omega')$. It is manifest that $r = r'_+$ is the first-order pole in the integral of Eq. (21). Doing the integral of r firstly, we could obtain

$$\mathrm{Im}S = (-\pi)\int_0^{\omega} \frac{2(m-\omega')^2 + 2(m-\omega')\sqrt{(m-\omega')^2 - a^2} - a^2}{\sqrt{(m-\omega')^2 - a^2}} d(-\omega') \tag{23}$$

At last, we could get

$$\mathrm{Im}S = \pi\left[m^2 - (m-\omega)^2 + m\sqrt{m^2 - a^2} - (m-\omega)\sqrt{(0m-\omega)^2 - a^2}\right] \tag{24}$$

The tunnelling rate is therefore

$$\Gamma \sim \exp(-2\,\mathrm{Im}S) = e^{\Delta S_{\rm BH}} \tag{25}$$

where $S_{\rm BH}$ is the Bekenstein–Hawking entropy.

4. CONCLUSION AND DISCUSSION

In this section, we will sum up the main points of the coordinates used to study the tunnelling effect.

First, the components of the metric in the coordinates should be regular at the event horizon and there should be a time-like killing vector field, such as the Painlevè and Eddington–Finkelstein coordinates. If the time-like killing vector field does not exist, the energy conservation would not be tenable. The condition that the constant-time slices should be the flat Euclidean space is not very important. The constant-time slices in line elements (1) and (14) are not the flat Euclidean space. The calculation in Sections 2 and 3, however, is shown that the WKB approximation is also tenable in the space which is not flat Euclidean.

Second, the event horizon and the time-like limit surface should coincide in the coordinate. We use the semi-classical WKB formula which is only tenable in the case of the infinite blue-shift near the horizon when we study the tunnelling process. In addition, if from $g_{00} = 0$ we could not obtain the event horizon, the first-order pole in Eq. (6) and (21) would be at the time-like limit surface $r = r_{\mathrm{TLS}}$, not at the event horizon, where r_{TLS} is the radius of the time-like limit surface. This means that the particle tunnels out of the time-like limit surface, not event horizon. It is well known, however, that Hawking radiation comes from the event horizon, not from the time-like limit surface.

Finally, in quantum mechanics, particle tunnelling a barrier is an instantaneous process. So, Zhang and Zhao (2005) suggests that the metric in the coordinates should satisfy Landau coordinate clock synchronization condition (Landau and Lifshitz, 1975). In fact, in the tunnelling process, after the particle tunnels out of the horizon, it is impossible for the particle to return to the place of departure–the place just inside the event horizon. The clocks staying along the radial direction could be always adjusted synchronously. So, we could study the tunnelling effect in the coordinates which do not satisfy the Landau coordinate clock synchronization condition, although the line elements of the Kerr black hole in Eddington–Finkelstein coordinates satisfy this condition.

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